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Two matroidal families on the edge set of a graph

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Abstract

Let G be a 2-connected undirected graph with n vertices. Its connected subgraphs of $n-1$ edges (that is, its spanning trees) are the bases of the usual cycle matroid of G. Let now X be a subset of vertices of G and consider those connected subgraphs of n edges whose unique circuit passes through at least one element of X. They are shown to be the bases of another matroid. A similar construction is given if the connectivity of the subgraph is not required but every circuit of the subgraph must pass through at least one element of X. Both constructions still lead to matroids if X is a subset of *edges* of G. Relation of the first construction to elementary strong maps (if G is planar) and representability properties of the matroids arising from these constructions are also presented. Finally, a civil engineering problem is described which served as the original motivation of this study. © 2002 Elsevier Science B.V. All rights reserved.

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Let G be a finite undirected 2-connected graph with vertex set V and edge-set E, let $n = |V|$ denote the number of its vertices. Those $n-1$ element subsets of E which correspond to connected (and hence circuit-free) subgraphs are called spanning trees and form the bases of a matroid $\mathscr{M}(G)$, also called the cycle matroid of G. (For definitions in graph and matroid theory see [5,8,10], for example.)

The main purpose of Section 1 is to prove the following observations:

Theorem 1. *Let $X \subseteq V(G)$ be an arbitrary nonempty subset of vertices.*

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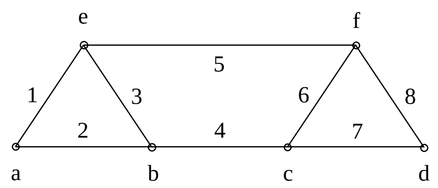

Fig. 1.

(a) *Consider the set* $\mathscr{B}'_X=\{B\colon B\subseteq E;\ |B|=n;$ *the subgraph of* G *determined by the edges of* B *is connected and the only circuit of it passes through at least one element of* $X\}$. *Then* $\mathscr{B}'_X$ *is the set of bases of a matroid* $\mathscr{M}'_X(G)$ *on the set* E.

(b) *Similarly*, $\mathscr{B}''_X$ *is the set of bases of another matroid* $\mathscr{M}''_X(G)$ *where* $\mathscr{B}''_X=\{B\colon B\subseteq E;\ |B|=n;$ *each connected component of the subgraph of* G *determined by the edges of* B *has a single circuit which passes through at least one element of* $X\}$.

For example, if G is the graph of Fig. 1 and $X=\{a,c\}$ then the set $\{1,2,3,4,5,7\}$ is a base in both matroids while $\{1,2,3,6,7,8\}$ is a base in the second matroid only. If X consists of a single element then the two matroids are clearly the same.

Proof. Recall that a nonempty collection $\mathscr{B}$ of subsets of E is the set of bases of a matroid on E if and only if each subset has the same cardinality and for any two subsets $B_1,B_2\in\mathscr{B}$ and for any element $x\in B_1$ there is an element $y\in B_2$ so that B_1-x+y is again a subset in $\mathscr{B}$. The equicardinality of the elements of $\mathscr{B}'_X$ and that of the elements of $\mathscr{B}''_X$ are obvious.

Case (a): Let $B_1\in\mathscr{B}'_X$ and let $x\in B_1$. If B_1-x is disconnected then we are done: since $B_2\in\mathscr{B}'_X$ determines a connected subgraph, at least one of its edges can play the role of y (and the circuit of B_1-x remains the circuit of B_1-x+y hence it passes through at least one element of X).

If B_1-x is connected (hence a spanning tree), then consider the only circuit C of B_2. Let $p\in X$ be a vertex of C and let q_1,q_2 be its two neighbours along C. If any of the edges $\{p,q_1\},\{p,q_2\}$ does not belong to B_1-x then it can play the role of y since then the unique circuit of B_1-x+y must contain the edge y, hence its vertex p as well.

Finally suppose that both $\{p,q_1\}$ and $\{p,q_2\}$ belong to the spanning tree B_1-x. If we delete these two edges, a 3-component forest remains. There exists at least one edge of C, different from $\{p,q_1\}$ and from $\{p,q_2\}$, which connects distinct components of this forest. Adding this edge to B_1-x leads to a circuit passing through p.

Case (b): Let $B_1\in\mathscr{B}''_X$. If the circuit-free component of B_1-x has vertex set V_1 and there is an edge in B_2 between V_1 and $V-V_1$ then this edge can play the role of y, as in the first paragraph of the proof of Case (a). Otherwise B_2 contains a circuit within V_1 and we proceed as in the second and third paragraphs of the proof of Case (a). $\square$

Remark 1. The 2-connectedness of our graph is not relevant. However, otherwise X must contain such vertices which belong to some circuits.

Theorem 2. *Let us assign arbitrary orientation to the edges of* G *and consider the* (*vertex-edge*) *incidence matrix* $\mathbf{V}$ *of* G. *In those rows which correspond to the vertices of* X, *replace the nonzero* (*that is*, ± 1) *elements by algebraically independent transcendental numbers. The new matrix* $\mathbf{V}_X$ *is a coordinatization of* $\mathscr{B}''_X$ *over the field* $\mathbb{R}$ *of the reals.*

Proof. It is well known that any row of $\mathbf{V}$ is the negative sum of the other $n-1$ rows and a subset of $n-1$ columns is linearly independent over the field $\mathbb{Q}$ of the rationals if and only if the corresponding edges form a spanning tree of G. Columns, corresponding to a circuit of G, are linearly dependent in $\mathbf{V}$ and they become linearly independent in $\mathbf{V}_X$ if and only if the circuit passes through at least one element of X. Hence n columns of $\mathbf{V}_X$, corresponding to a subset Y of E, are linearly independent over $\mathbb{R}$ if and only if $Y \in \mathscr{B}''_X$. $\square$

This proof can be considered as an alternative proof for statement (b) of Theorem 1 as well.

As a byproduct of Theorem 2 we obtain

Corollary 3. $\mathscr{M}''_X(G)$ *is representable over fields of sufficiently large cardinality*, *for any* G *and for any* $X \subseteq V(G)$.

This is a very modest result since $\mathscr{M}(G)$, like every graphic matroid, is regular, that is, representable over any field. However, the representability properties of these new matroids are not very good; in fact the above observation and also Corollary 6 below are best possible in the following sense:

Theorem 4. *For any positive integer* t *there exists a graph* G_t *and a vertex* p *in* $V(G_t)$ *so that* $\mathscr{M}'_{\{p\}}(G_t)$ *and* $\mathscr{M}''_{\{p\}}(G_t)$ *are representable over fields of cardinality larger than* t *only.*

Proof. Observe that if G_t consists of two vertices, p and q, and $t+2$ parallel edges only then $\mathscr{M}'_{\{p\}}(G_t)$ and $\mathscr{M}''_{\{p\}}(G_t)$ are isomorphic to the rank two uniform matroid $\mathscr{U}_{t+2,2}$ on $t+2$ elements. Then the statement trivially follows (cf. [4] as well). $\square$

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Let $\mathrm{Star}(v)$ denote the set of edges, incident to a vertex v, and let $\mathrm{Star}(X) = \bigcup_{x \in X} \mathrm{Star}(x)$ if X is a subset of vertices. Since a circuit passes through at least one vertex of X if and only if it contains at least one (in fact, at least 2, see Remark 2) edge of Star(X), the following theorem is clearly a generalization of Theorem 1.

Theorem 5. *Statements* (a) *and* (b) *of Theorem* 1 *remain true if* $X \subseteq E(G)$ *is a nonempty subset of edges.*

Proof. Case (a) has a very simple proof, due to Szigeti [9]. Define a rank one matroid $\mathcal{N}_X$ on the edge set $E(G)$ of the graph G as follows: an element $e \in E(G)$ is a base in $\mathcal{N}_X$ if and only if $e \in X$. Then obviously $\mathcal{M}'_X(G)$ is the sum of $\mathcal{M}(G)$ and $\mathcal{N}_X$.

Case (b): Follow the proof of Theorem 2 with a single difference: replace the nonzero elements of the *columns* of **V**, corresponding to the edges of X, by algebraically independent transcendental numbers. □

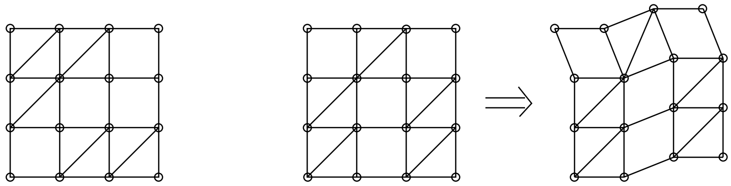
Remark 2. If $X \subseteq E$ happens to be the union of some stars then each circuit intersecting X has at least two common edges with it. However, for edge subsets in general, we cannot obtain a matroid if we require for the independence that the subgraph should be either circuit-free or its only circuit passes through at least two edges of X. For example, let G be a circuit formed by the edges $\{1,2,3,4\}$ (in this order) plus a diagonal edge 5 forming triangles $\{1,2,5\}$, $\{3,4,5\}$. If $X = \{1,3\}$ then the subsets $\{1,2,3,4\}$, $\{2,4,5\}$ would violate the exchange axiom for the independent sets.

Remark 3. The statement and the proof of Theorem 5(a) is valid for any matroid instead of $\mathcal{M}(G)$ and those of Theorem 5(b) for any matroid representable over $\mathbb{R}$. The first statement is trivial; for the second consider a matrix representation over $\mathbb{R}$. Without loss of generality we may suppose that the rows of the matrix are linearly independent. Extend the matrix with an additional row, obtained as the negative sum of all the original rows. Finally, in those columns of the extended matrix which correspond to the elements of X, replace the nonzero elements by algebraically independent transcendentals.

The sum of two regular matroids is known to be representable over fields of sufficiently large cardinality, see [6], hence the proof of Theorem 5(a) immediately leads to

Corollary 6. $\mathcal{M}'_X(G)$ *is representable over fields of sufficiently large cardinality*, *for any* G *and for any* $X \subseteq E(G)$.

Combining this idea with that of Edmonds [2] we can directly give an algebraic construction for the representation of $\mathcal{M}'_X(G)$, similar to that of Theorem 2. Start with the matrix **V** again and add an extra row, consisting of zeroes for the columns not belonging to X and algebraically independent transcendentals for the columns belonging to X.



(a) (b) (c)

Fig. 2.

(a) (b)

Fig. 3.

3

As an “application” of these results we present a theorem on the rigidity of some special 2-dimensional bar-and-joint frameworks. This was the original motivation of this study. For the general solution of this engineering problem the reader is referred to [3].

Consider a $k \times l$ square grid as a planar framework composed of rigid bars and flexible joints. If diagonal rods are added to some squares then the resulting framework may become rigid (Fig. 2a) or may remain nonrigid (Figs. 2b and c).

Deformations like on Fig. 2c may be obtained by successive deformations of whole rows or columns of squares and the presence of a diagonal in a square only means that the magnitude of the deformations were the same for the row and for the column of that square. Hence if the rows and the columns correspond to the vertices of a bipartite graph and the diagonals correspond to edges of this graph in a straightforward way (see Figs. 3a and b) then the framework is rigid if and only if the corresponding bipartite graph is connected [1].

In particular, for a $k \times 3$ square grid we need $k+2$ diagonals in an appropriate way. If we remove an original bar (see Fig. 4a), we can expect that, among the remaining $3k-2$ squares, a suitable set of $k+3$ must be extended by diagonals to make the framework rigid. (The infinitesimal motion, indicated by dotted lines in Fig. 4b, shows that the “vertical” rod must be removed from the second row—otherwise the deformation cannot be prevented by diagonals of the remaining squares.)

Theorem 7. *Let $K_{3,k}$ be a complete bipartite graph with vertex set $\{p_1, p_2, p_3, q_1, q_2, \dots, q_k\}$ and remove the edges $\{p_2, q_i\}$ and $\{p_2, q_{i+1}\}$ for some $2 \leqslant i \leqslant k-2$. Denote the resulting graph with G and the set $\{q_i, q_{i+1}\}$ by X. Then a set of $k+3$*

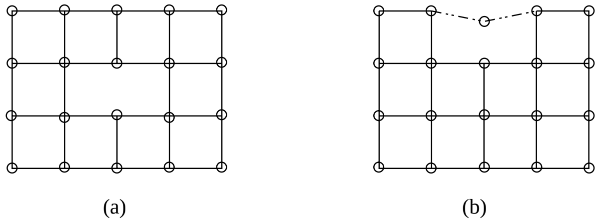


Fig. 4.

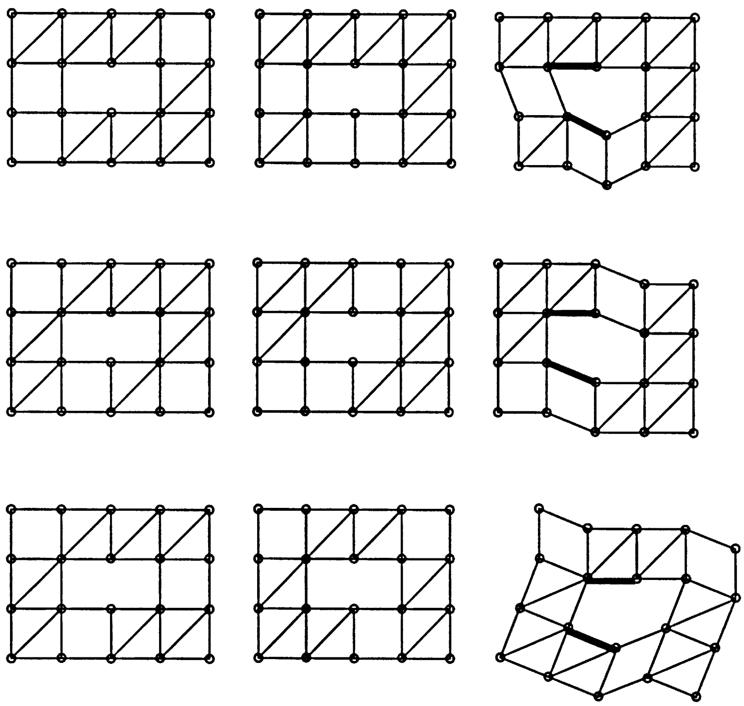


Fig. 5.

diagonals makes the above framework (*where the vertical bar in the second row between rooms i and $i+1$ is removed*) *rigid if and only if the corresponding edge set is a base in $\mathscr{M}'_X(G)$.*

Example. Let $k=4$, as in Fig. 4a. The first column of Fig. 5 shows some appropriate bracings where the circuit passes through both q_2 and q_3 (top) or through q_3 only (middle and bottom). The second column contains bracings where the circuits avoid q_2 and q_3. Accordingly, the planar frameworks in the first column are rigid (although this might not look quite obvious at the first glance) while those in the second column are not (see deformations in the third column). Observe that during these deformations some bars (indicated by heavy lines) do not remain parallel due to the lack of the "central" vertical bar.

4

In this section, we show that $\mathscr{M}'_X(G)$, even in the more general case when $X \subseteq E(G)$, can be constructed in a previously known way if G is planar (which is the case in the above civil engineering application).

Suppose that the vertex set $\{v_1, v_2, \ldots, v_n\}$ of a graph is weighted by some weights $w_1, w_2, \ldots, w_n$ from an arbitrary field F, satisfying $\sum_{i=1}^{n} w_i = 0$. A 2-component forest of the graph is called *asymmetric* if the weight sums within the components are different from 0. Then the asymmetric forests form the bases of a matroid which is a strong map of the cycle matroid of this graph, see [7].

Let G be a planar graph and $X \subseteq E(G)$. Consider a planar representation of G, construct its dual G^*, and define the following equivalence relation ρ_X on the elements of $V(G^*)$. Two elements a, b are in relation if and only if either $a = b$ or there exists a path in G^* between a and b using edges of X only. Now ρ_X defines a partition of $V(G^*)$ consisting of possibly some single elements and subsets

$$\{v_{1,1}, v_{1,2}, \ldots\}, \{v_{2,1}, v_{2,2}, \ldots\}, \{v_{k,1}, v_{k,2}, \ldots\}$$

each of cardinality at least two. Finally define a real valued weight function $w: V(G^*) \to \mathbb{R}$ so that

$$\begin{aligned}
&\{w(v_{i,j});\ 1 \leqslant i \leqslant k,\ j \geqslant 2\} && \text{be algebraically independent over } \mathbb{Q}, \\
&w(v_{i,1}) = -\sum_{j \geqslant 2} w(v_{i,j}) && \text{and} \\
&w(v) = 0 && \text{for vertices not included in any of these subsets.}
\end{aligned}$$

Theorem 8. *$\mathscr{M}'_X(G)$ is just the dual of the elementary strong map of $\mathscr{M}(G^*)$ determined by this weight function w.*

Proof. A 2-component forest F of G^* is asymmetric with respect to this weight function w if and only if the cut set formed by edges connecting the two components intersects at least one of the subsets $\{v_{i,1}, v_{i,2}, \ldots\}$. By the definition of ρ_X this happens if and only if there exists at least one edge of X with endpoints in different components of F. Thus the bases of the dual are such subgraphs of n edges in G where the unique circuit passes through at least one edge of X. $\square$

For example, if G is the graph of Fig. 1 and $X = \{2, 3, 6\}$ then X spans G^*, hence all the weights in $V(G^*)$ are algebraically independent and the elementary strong map is just the truncation of $\mathscr{M}(G^*)$ (that is, every 2-component forest in $\mathscr{M}(G^*)$ is asymmetric). Thus every non-parallel pair of edges are bases in the truncation, hence every subgraph of six edges is a base of $\mathscr{M}'_X(G)$ except the complements of $\{1,2\}$, $\{4,5\}$ or $\{7,8\}$.

On the other hand, if $X=\{2,7\}$ then the vertex of $V(G^*)$, determined by the face $\{3,4,6,5\}$ will receive a zero-weight. Accordingly, the pairs $\{1,7\},\{1,8\},\{2,7\},\{2,8\}$ are symmetric 2-component forests of $\mathscr{M}(G^*)$. Hence $\{3,4,5,6\}$ extended by one of these pairs is a base in $\mathscr{M}'_{\{2,3,6\}}(G)$ but not in $\mathscr{M}'_{\{2,7\}}(G)$.

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